Problem 1.28

Go over the steps from Equation (1.25) to (1.29) in the proof of conservation of momentum, but treat the case that $N = 3$ and write out all the summations explicitly to be sure you understand the various manipulations.

Solution

Suppose there are N particles in a system. One of the particles, labelled α, experiences a force from the remaining $N-1$ particles in addition to an external force.

$$\text{Net Force on Particle } \alpha = \mathbf{F}_\alpha = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{N} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}}$$

If $N = 3$, then

$$\mathbf{F}_\alpha = \sum_{\substack{\beta=1 \\ \beta\neq\alpha}}^{3} \mathbf{F}_{\alpha\beta} + \mathbf{F}_\alpha^{\text{ext}} \quad \Rightarrow \quad \begin{cases} \mathbf{F}_1 = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{\text{ext}} \\ \\ \mathbf{F}_2 = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_2^{\text{ext}} \\ \\ \mathbf{F}_3 = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_3^{\text{ext}} \end{cases}.$$

According to Newton's second law, the net force on a particle is equal to the rate of change of momentum.

$$\begin{cases} \dfrac{d\mathbf{p}_1}{dt} = \mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{\text{ext}} \\ \\ \dfrac{d\mathbf{p}_2}{dt} = \mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_2^{\text{ext}} \\ \\ \dfrac{d\mathbf{p}_3}{dt} = \mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_3^{\text{ext}} \end{cases}$$

Add them together to get the rate of change of total momentum. By Newton's third law, $\mathbf{F}_{ij} + \mathbf{F}_{ji} = \mathbf{0}$.

$$\begin{aligned} \frac{d\mathbf{P}}{dt} &= \sum_{\beta=1}^{3} \frac{d\mathbf{p}_\beta}{dt} \\ &= \left(\mathbf{F}_{12} + \mathbf{F}_{13} + \mathbf{F}_1^{\text{ext}}\right) + \left(\mathbf{F}_{21} + \mathbf{F}_{23} + \mathbf{F}_2^{\text{ext}}\right) + \left(\mathbf{F}_{31} + \mathbf{F}_{32} + \mathbf{F}_3^{\text{ext}}\right) \\ &= \underbrace{(\mathbf{F}_{12} + \mathbf{F}_{21})}_{=\,\mathbf{0}} + \underbrace{(\mathbf{F}_{13} + \mathbf{F}_{31})}_{=\,\mathbf{0}} + \underbrace{(\mathbf{F}_{23} + \mathbf{F}_{32})}_{=\,\mathbf{0}} + \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} \\ &= \mathbf{F}_1^{\text{ext}} + \mathbf{F}_2^{\text{ext}} + \mathbf{F}_3^{\text{ext}} \end{aligned}$$

Therefore, if the net external force acting on the particles is zero,

$$\frac{d\mathbf{P}}{dt} = \mathbf{0},$$

then the total momentum is conserved: $\mathbf{P}_{\text{initial}} = \mathbf{P}_{\text{final}} = \text{constant}$.